Chapter 7. Dynamics of Systems of Particles

(Most of the material presented in this chapter is taken from Thornton and Marion, Chap. 9.)

In this chapter we study the dynamics of systems composed potentially of a large number of particles, and inquire on conservation theorems and the behavior of systems that exhibit mass loss (e.g., rockets).

7.1 Centre of Mass

For a system composed of n particles, the total mass M is given by

$$M = \sum_{\alpha} m_{\alpha}, \tag{7.1}$$

where m_{α} is the mass of the αth particle, with $\alpha = 1, \ldots, n$. If each particle is (mathematically) connected to the origin of the system through a position vector $\mathbf{r}_{\alpha}$, then the centre of mass vector is defined as

$$\mathbf{R} = \frac{1}{M} \sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha}. \tag{7.2}$$

For a continuous system, the summation over α is replaced with an integral over an infinitesimal amount of mass dm such that

$$\mathbf{R} = \frac{1}{M} \int \mathbf{r}\, dm. \tag{7.3}$$

It is important to realize that the position vector $\mathbf{R}$ of the centre of mass depends on the origin chosen for the coordinate systems.

7.2 The Conservation of Linear Momentum

The force acting on particle α of a system of particles is composed of the resultant of all forces external to the system $\mathbf{F}_{\alpha}^{(e)}$, and the resultant of the internal forces $\mathbf{f}_{\alpha}$ stemming from its interaction with the other particles that are part of the system. If we define these internal interaction forces as $\mathbf{f}_{\alpha\beta}$, the resulting force $\mathbf{f}_{\alpha}$ acting on particle α is

$$\mathbf{f}_{\alpha} = \sum_{\beta \neq \alpha} \mathbf{f}_{\alpha\beta}. \tag{7.4}$$

The total force $\mathbf{F}_{\alpha}$ acting on the particle is

$$\mathbf{F}_\alpha = \mathbf{F}_\alpha^{(e)} + \mathbf{f}_\alpha. \tag{7.5}$$

From Newton's Second Law we can write

$$\dot{\mathbf{p}}_\alpha = \mathbf{F}_\alpha^{(e)} + \mathbf{f}_\alpha, \tag{7.6}$$

or

$$\begin{aligned} \frac{d^2}{dt^2}\left(m_\alpha \mathbf{r}_\alpha\right) &= \mathbf{F}_\alpha^{(e)} + \mathbf{f}_\alpha \\ &= \mathbf{F}_\alpha^{(e)} + \sum_{\beta \neq \alpha} \mathbf{f}_{\alpha\beta}, \end{aligned} \tag{7.7}$$

where no summation on repeated index is implied.

Summing equation (7.7) over all particles we get

$$\begin{aligned} \frac{d^2}{dt^2}\left(\sum_\alpha m_\alpha \mathbf{r}_\alpha\right) &= \sum_\alpha \mathbf{F}_\alpha^{(e)} + \sum_\alpha \sum_{\beta \neq \alpha} \mathbf{f}_{\alpha\beta} \\ &= \mathbf{F} + \sum_{\alpha,\beta\ pairs} \left(\mathbf{f}_{\alpha\beta} + \mathbf{f}_{\beta\alpha}\right), \end{aligned} \tag{7.8}$$

where we have defined the sum over all external forces as

$$\mathbf{F} \equiv \sum_\alpha \mathbf{F}_\alpha^{(e)}, \tag{7.9}$$

and the second term on the right of equation (7.8) was replaced by a single summation over every pair of internal interactions between the particles. However, we know from Newton's Third Law that $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$. We can therefore write, from equation (7.8) that

$$M\ddot{\mathbf{R}} = \mathbf{F}. \tag{7.10}$$

This last equation can also be used to express the conservation of momentum since

$$\mathbf{P} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha = \frac{d}{dt}\left(\sum_\alpha m_\alpha \mathbf{r}_\alpha\right) = M\dot{\mathbf{R}}, \tag{7.11}$$

and then

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F}. \tag{7.12}$$

We can summarize this result as follows

I. *The centre of mass of a system moves as if it were a single particle of mass equal to the total mass of the system, acted upon by the total external force, and independent of the internal forces (as long as* $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$ *(Newton's Third Law) holds).*

II. *The total linear momentum of a system is the same as that of a singe particle of mass* M *located at the position of the centre of mass and moving in the manner the centre of mass moves.*

III. *The total linear momentum for a system free of external forces is a constant and equal to the linear momentum of the centre of mass* (the law of conservation of linear momentum for a system).

7.3 The Conservation of Angular Momentum

As we saw in the previous chapter on central force motion, it is often more convenient to define the positions of the particles composing a system by vectors $\mathbf{r}'_\alpha$ originating at the centre of mass (see Figure 7-1). The position vector $\mathbf{r}_\alpha$ in the inertial frame is

$$\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha. \tag{7.13}$$

The angular momentum of the αth particle is given by

$$\mathbf{L}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha, \tag{7.14}$$

and summing over all particles

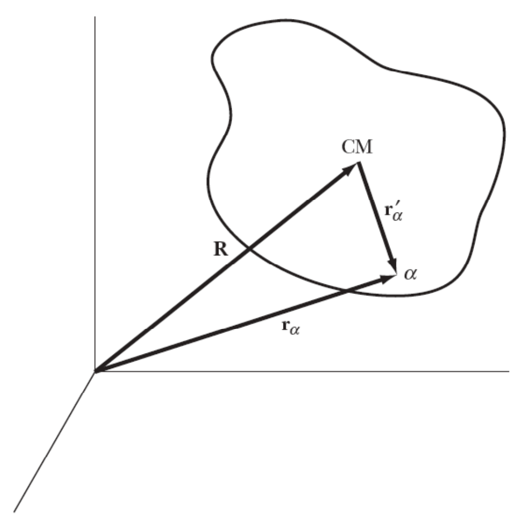

Figure 7-1 – Description of the position of a particle using its position vector from the centre of mass of the system.

$$
\begin{aligned}
\mathbf{L} &= \sum_\alpha \mathbf{L}_\alpha = \sum_\alpha \left(\mathbf{r}_\alpha \times \mathbf{p}_\alpha\right) = \sum_\alpha \left(\mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha\right) \\
&= \sum_\alpha \left[\left(\mathbf{R} + \mathbf{r}'_\alpha\right) \times m_\alpha \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha\right)\right] \\
&= \sum_\alpha m_\alpha \left[\left(\mathbf{R} \times \dot{\mathbf{R}}\right) + \left(\mathbf{R} \times \dot{\mathbf{r}}_\alpha{}'\right) + \left(\mathbf{r}'_\alpha \times \dot{\mathbf{R}}\right) + \left(\mathbf{r}'_\alpha \times \dot{\mathbf{r}}'_\alpha\right)\right].
\end{aligned}
\tag{7.15}
$$

The second and third terms on the right hand side equal zero from

$$
\begin{aligned}
\sum_\alpha m_\alpha \left[\left(\mathbf{R} \times \dot{\mathbf{r}}'_\alpha\right) + \left(\mathbf{r}'_\alpha \times \dot{\mathbf{R}}\right)\right] &= \sum_\alpha m_\alpha \left[\frac{d}{dt}\left(\mathbf{R} \times \mathbf{r}'_\alpha\right) - \left(\dot{\mathbf{R}} \times \mathbf{r}'_\alpha\right) + \left(\mathbf{r}'_\alpha \times \dot{\mathbf{R}}\right)\right] \\
&= \frac{d}{dt}\left(\mathbf{R} \times \sum_\alpha m_\alpha \mathbf{r}'_\alpha\right) + 2\left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha\right) \times \mathbf{R} = 0,
\end{aligned}
\tag{7.16}
$$

since, from equations (7.2) and (7.13),

$$
\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0. \tag{7.17}
$$

Equation (7.15) now becomes

$$
\begin{aligned}
\mathbf{L} &= \left(\mathbf{R} \times \sum_\alpha m_\alpha \dot{\mathbf{R}}\right) + \sum_\alpha \left(\mathbf{r}'_\alpha \times m_\alpha \dot{\mathbf{r}}'_\alpha\right) \\
&= \mathbf{R} \times \mathbf{P} + \sum_\alpha \left(\mathbf{r}'_\alpha \times \mathbf{p}'_\alpha\right).
\end{aligned}
\tag{7.18}
$$

We, therefore, have this important result

IV. *The angular momentum about an origin is the sum of the angular momentum of the centre of mass about that origin and the angular momentum of the system about the position of the centre of mass.*

The time derivative of the total angular momentum is

$$
\begin{aligned}
\dot{\mathbf{L}} &= \sum_\alpha \dot{\mathbf{L}}_\alpha = \sum_\alpha \left(\mathbf{r}_\alpha \times \dot{\mathbf{p}}_\alpha\right) \\
&= \sum_\alpha \left(\mathbf{r}_\alpha \times \mathbf{F}_\alpha^{(e)}\right) + \sum_\alpha \left(\mathbf{r}_\alpha \times \sum_{\beta \neq \alpha} \mathbf{f}_{\alpha\beta}\right) \\
&= \sum_\alpha \left(\mathbf{r}_\alpha \times \mathbf{F}_\alpha^{(e)}\right) + \sum_{\alpha<\beta} \left[\left(\mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta}\right) + \left(\mathbf{r}_\beta \times \mathbf{f}_{\beta\alpha}\right)\right],
\end{aligned}
\tag{7.19}
$$

where $\sum\limits_{\alpha<\beta}$ means a sum over α and β with $\alpha < \beta$.

We know, however, from Newton's Third Law that $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$ so that equation (7.19) can be re-written

$$\dot{\mathbf{L}} = \sum_{\alpha} \left(\mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(e)} \right) + \sum_{\alpha<\beta} \left[\left(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta} \right) \times \mathbf{f}_{\alpha\beta} \right]. \tag{7.20}$$

If we further limit ourselves to internal forces $\mathbf{f}_{\alpha\beta}$ that are also directed along the straight line joining the two interacting particles (i.e., along $\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}$), we must have the following

$$\left(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta} \right) \times \mathbf{f}_{\alpha\beta} = 0. \tag{7.21}$$

The time derivative of the total angular momentum is then

$$\dot{\mathbf{L}} = \sum_{\alpha} \left(\mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{(e)} \right), \tag{7.22}$$

or if we express the right hand side as a sum of the external torque applied on the different particles $\mathbf{N}_{\alpha}^{(e)}$

$$\dot{\mathbf{L}} = \sum_{\alpha} \mathbf{N}_{\alpha}^{(e)} = \mathbf{N}^{(e)}. \tag{7.23}$$

We, therefore, have the following results

V. *If the net resultant external torque about an axis vanishes, then the total angular momentum of the system about that axis remains a constant in time.*

Furthermore, since we found that the total internal torque also vanishes, i.e.,

$$\sum_{\alpha} \left(\mathbf{r}_{\alpha} \times \sum_{\beta\neq\alpha} \mathbf{f}_{\alpha\beta} \right) = \sum_{\alpha<\beta} \left[\left(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta} \right) \times \mathbf{f}_{\alpha\beta} \right] = 0, \tag{7.24}$$

and we can state that

VI. *The total internal torque must vanish if the internal forces are central (i.e., $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$ and the internal forces between two interacting particles are directed along the line joining them), and the angular momentum of an isolated system cannot be altered without the application of external forces.*

7.4 The Energy of the System

Consider a system of particles that evolves from a starting configuration "1" to an ulterior configuration "2" where the positions $\mathbf{r}_{\alpha}$ of the particles may have changed in the process. We can write the total work done on the system as the sum of the work done on individual particles

$$
\begin{aligned}
W_{12} &= \sum_\alpha \int_1^2 \mathbf{F}_\alpha \bullet d\mathbf{r}_\alpha \\
&= \sum_\alpha \int_1^2 m_\alpha \frac{d\mathbf{v}_\alpha}{dt} \bullet \frac{d\mathbf{r}_\alpha}{dt} dt = \sum_\alpha \int_1^2 m_\alpha \frac{d\mathbf{v}_\alpha}{dt} \bullet \mathbf{v}_\alpha dt \\
&= \sum_\alpha \int_1^2 \frac{1}{2} m_\alpha \frac{dv_\alpha{}^2}{dt} dt = \sum_\alpha \int_1^2 \frac{d}{dt}\left(\frac{1}{2} m_\alpha v_\alpha{}^2\right) dt \\
&= \sum_\alpha \int_1^2 d\left(\frac{1}{2} m_\alpha v_\alpha{}^2\right) = T_2 - T_1,
\end{aligned}
\tag{7.25}
$$

where

$$
T = \sum_\alpha T_\alpha = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha{}^2. \tag{7.26}
$$

Using equation (7.13) we can write

$$
\begin{aligned}
v_\alpha{}^2 = \dot{\mathbf{r}}_\alpha \bullet \dot{\mathbf{r}}_\alpha &= \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha\right) \bullet \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}'_\alpha\right) \\
&= \dot{\mathbf{R}} \bullet \dot{\mathbf{R}} + 2\left(\dot{\mathbf{r}}'_\alpha \bullet \dot{\mathbf{R}}\right) + \left(\dot{\mathbf{r}}'_\alpha \bullet \dot{\mathbf{r}}'_\alpha\right) \\
&= V^2 + 2\left(\dot{\mathbf{r}}'_\alpha \bullet \dot{\mathbf{R}}\right) + \left(v'_\alpha\right)^2,
\end{aligned}
\tag{7.27}
$$

where $v_\alpha{}' = \left|\dot{\mathbf{r}}'_\alpha\right|$ and $V = \left|\dot{\mathbf{R}}\right|$. Inserting equation (7.27) into equation (7.26), while using the earlier result that states that $\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0$, we find that

$$
T = \frac{1}{2} M V^2 + \sum_\alpha \frac{1}{2} m_\alpha \left(v'_\alpha\right)^2. \tag{7.28}
$$

In other words

VII. *The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with velocity of the centre of mass and the kinetic energy of motion of the individual particles relative to the centre of mass.*

Alternatively, we can rewrite first of equations (7.25) by separating the total force applied on each particle in its external and internal components

$$W_{12} = \sum_{\alpha} \int_1^2 \mathbf{F}_{\alpha}^{(e)} \bullet d\mathbf{r}_{\alpha} + \sum_{\alpha,\beta\neq\alpha} \int_1^2 \mathbf{f}_{\alpha\beta} \bullet d\mathbf{r}_{\alpha}. \tag{7.29}$$

If the forces involved are conservatives, we can then derive them from potentials such that

$$\begin{aligned} \mathbf{F}_{\alpha}^{(e)} &= -\nabla_{\alpha} U_{\alpha} \\ \mathbf{f}_{\alpha\beta} &= -\nabla_{\alpha} \bar{U}_{\alpha\beta}, \end{aligned} \tag{7.30}$$

where U_{α} and $\bar{U}_{\alpha\beta}$ are independent potential functions. The gradient operator ∇_{α} is a vector operator meant to apply to the coordinate components of the αth particle (i.e., α is the index that specifies a given particle, and does not represent a coordinate such as x, y, or z).

The first term on the right hand side of equation (7.29) can be written as

$$\begin{aligned} \sum_{\alpha} \int_1^2 \mathbf{F}_{\alpha}^{(e)} \bullet d\mathbf{r}_{\alpha} &= -\sum_{\alpha} \int_1^2 \left(\nabla_{\alpha} U_{\alpha} \right) \bullet d\mathbf{r}_{\alpha} \\ &= -\sum_{\alpha} U_{\alpha} \Big|_1^2 . \end{aligned} \tag{7.31}$$

The last term of the same equation is transformed to

$$\begin{aligned} \sum_{\alpha,\beta\neq\alpha} \int_1^2 \mathbf{f}_{\alpha\beta} \bullet d\mathbf{r}_{\alpha} &= \sum_{\alpha<\beta} \int_1^2 \left(\mathbf{f}_{\alpha\beta} \bullet d\mathbf{r}_{\alpha} + \mathbf{f}_{\beta\alpha} \bullet d\mathbf{r}_{\beta} \right) \\ &= \sum_{\alpha<\beta} \int_1^2 \mathbf{f}_{\alpha\beta} \bullet \left(d\mathbf{r}_{\alpha} - d\mathbf{r}_{\beta} \right). \end{aligned} \tag{7.32}$$

Before we use the last of equations (7.30) to further transform equation (7.32), we consider the following differential

$$\begin{aligned} d\bar{U}_{\alpha\beta} &= \left(\nabla_{\alpha} \bar{U}_{\alpha\beta} \right) \bullet d\mathbf{r}_{\alpha} + \left(\nabla_{\beta} \bar{U}_{\beta\alpha} \right) \bullet d\mathbf{r}_{\beta} \\ &= \left(-\mathbf{f}_{\alpha\beta} \right) \bullet d\mathbf{r}_{\alpha} + \left(-\mathbf{f}_{\beta\alpha} \right) \bullet d\mathbf{r}_{\beta} \\ &= -\mathbf{f}_{\alpha\beta} \bullet \left(d\mathbf{r}_{\alpha} - d\mathbf{r}_{\beta} \right), \end{aligned} \tag{7.33}$$

since $\nabla_{\beta} \bar{U}_{\beta\alpha} = -\mathbf{f}_{\beta\alpha} = \mathbf{f}_{\alpha\beta}$ (note also that $\bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha}$). Combining this result with equations (7.29), (7.31), and (7.32), we get

$$W_{12} = -\sum_{\alpha} U_\alpha \Big|_1^2 - \sum_{\alpha<\beta} \bar{U}_{\alpha\beta} \Big|_1^2 . \tag{7.34}$$

If we define the total potential energy as

$$U = U_\alpha + \bar{U}_{\alpha\beta}, \tag{7.35}$$

we get

$$W_{12} = -U\big|_1^2 = U_1 - U_2. \tag{7.36}$$

Combining equation (7.36) and the last of equations (7.25), we find that

$$T_2 - T_1 = U_1 - U_2 \tag{7.37}$$

or,

$$T_1 + U_1 = T_2 + U_2, \tag{7.38}$$

and finally

$$E_1 = E_2. \tag{7.39}$$

We have therefore proved the conservation of energy for a system of particles where all the forces can be derived from a potential that are independent of time; such a system is called *conservative*.

VIII. *The total energy for a conservative system is constant.*

7.5 Rocket Motion

We now work out two examples dealing the motion of rockets. The first one concerns a rocket in free space, whereas the second deals with vertical ascent under gravity.

7.5.1 Rocket Motion in Free Space

We consider the case where a rocket is moving under the influence of no external forces. We also choose a closed system where, therefore, the linear momentum will be conserved. We assume that the rocket is moving in an inertial reference frame in the x direction at velocity $\mathbf{v} = v\mathbf{e}_x$. During a infinitesimal time interval dt an infinitesimal amount of mass dm' is ejected from the rocket engine with a speed $\mathbf{u} = -u\mathbf{e}_x$ with respect to the ship (see Figure 7-2). If we define the quantities $\mathbf{p}(t) = p(t)\mathbf{e}_x$ as the momentum at time t, we can write

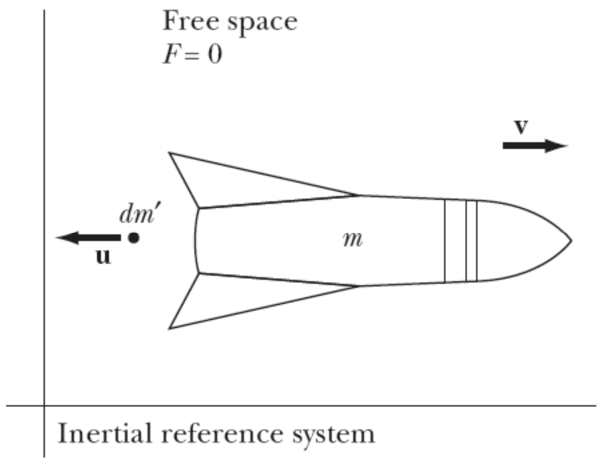

Figure 7-2 – A rocket moves in free space at velocity $\mathbf{v}$. In the time interval dt**,** a mass dm' is ejected from the rocket engine with velocity $\mathbf{u}$ with respect to the rocket ship.

$$\begin{aligned} p(t) &= mv \\ p(t+dt) &= (m-dm')(v+dv)+dm'(v-u). \end{aligned} \tag{7.40}$$

Since we must have conservation of linear momentum (because there are no external forces acting on the rocket), we have

$$\begin{aligned} p(t) &= p(t+dt) \\ mv &= (m-dm')(v+dv)+dm'(v-u) \\ mv &= mv+m\,dv-dm'v-dm'dv+dm'v-u\,dm', \end{aligned} \tag{7.41}$$

and

$$m\,dv = u\,dm', \tag{7.42}$$

or

$$dv = u\frac{dm'}{m}. \tag{7.43}$$

In going from the last of equations (7.41) to equation (7.42) we have neglected the term $dm'dv$, which is a second order term. The (positive) amount of mass dm' ejected from the rocket is, obviously, equal to the amount of mass *lost* by the ship. We can then write

$$dm = -dm', \tag{7.44}$$

and

$$dv = -u\frac{dm}{m}. \tag{7.45}$$

If m_0 and v_0 are, respectively, the initial mass and speed of the rocket we can integrate equation (7.45) to yield

$$\begin{aligned}
\int_{v_0}^{v} dv &= -u\int_{m_0}^{m}\frac{dm}{m} \\
v - v_0 &= u\ln\left(\frac{m_0}{m}\right) \\
v &= v_0 + u\ln\left(\frac{m_0}{m}\right),
\end{aligned} \tag{7.46}$$

where the exhaust velocity u was assumed to be a constant (i.e., not a function of v or m). Thus to maximize the speed of the rocket, one needs to maximize the exhaust velocity u and the ratio m_0/m. This is the reason why engineers have conceived multistage rockets, where independent fuel containers can be jettisoned when they empty.

For example, a multistage rocket might have an initial mass m_0, while its mass after the so-called "first-stage fuel container" has emptied is

$$m_1 = m_a + m_b, \tag{7.47}$$

where m_a and m_b are the mass of the first-stage payload and first-stage fuel container, respectively. We can express the terminal speed v_1 reached by the rocket after all the fuel of the first-stage fuel container has burnt out with the last of equations (7.46)

$$v_1 = v_0 + u\ln\left(\frac{m_0}{m_1}\right). \tag{7.48}$$

At that time, the mass m_b of the first-stage fuel container is released into space, and the second-stage rocket ignites. We now have m_a for the starting mass of space ship (second stage), and m_2 for its mass after the second-stage fuel container has burnt out. The terminal velocity is given by

$$\begin{aligned}
v_2 &= v_1 + u\ln\left(\frac{m_a}{m_2}\right) \\
&= v_0 + u\ln\left(\frac{m_0}{m_1}\right) + u\ln\left(\frac{m_a}{m_2}\right) \\
&= v_0 + u\ln\left(\frac{m_0 m_a}{m_1 m_2}\right).
\end{aligned} \tag{7.49}$$

The term $\left(m_0 m_a / m_1 m_2\right)$ can be made much larger than $\left(m_0 / m_1\right)$.

Engineers and scientists usually give the following definition to the commonly used force term "thrust"

$$\text{Thrust} \equiv -u\frac{dm}{dt}, \tag{7.50}$$

which is greater than zero since $dm/dt < 0$.

7.5.2 **Vertical Ascent Under Gravity**

We now consider the case of a rocket that is attempting to break free from the Earth's gravitational pull. In order for the problem to be tractable analytically, we will assume that the rocket has only a vertical motion (i.e., no horizontal movement), there is no air resistance, and the gravitational field is constant with height. From Figure 7-3 we see that

$$\begin{aligned} \mathbf{v} &= v\mathbf{e}_y \\ \mathbf{g} &= -g\mathbf{e}_y, \end{aligned} \tag{7.51}$$

and similar equations for the other quantities involved in the problem.

We can use the results of the previous case of motion in free space, but we no longer have $\mathbf{F}^{(e)} = 0$. As before, the ejected mass is given by $dm' = -dm$. From Newton's Second Law, the external force is

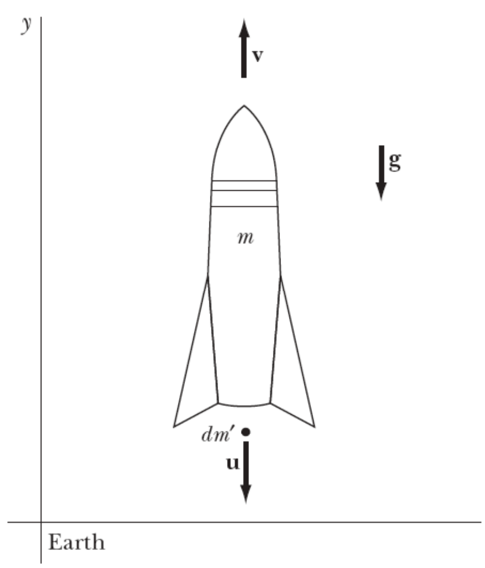

Figure 7-3 – A rocket in vertical ascent under Earth's gravity. A mass dm' is ejected from the rocket engine, during a time interval dt, with a velocity $\mathbf{u}$ with respect to the rocket ship.

$$F^{(e)} = \frac{dp}{dt} = \frac{d}{dt}(mv), \tag{7.52}$$

or

$$\begin{aligned} F^{(e)}dt &= dp = p(t+dt) - p(t) \\ &= (m - dm')(v + dv) + (v - u)\,dm' - mv \\ &= m\,dv + u\,dm, \end{aligned} \tag{7.53}$$

where we have neglected any second order terms. Dividing this result by dt we find

$$F^{(e)} = -mg = m\dot{v} + u\dot{m}. \tag{7.54}$$

We can manipulate equation (7.54) to isolate dv

$$\begin{aligned} dv &= -\left(g + \frac{u}{m}\frac{dm}{dt}\right)dt \\ &= -g\,dt - u\frac{dm}{m}. \end{aligned} \tag{7.55}$$

Integration of this last equation yields

$$v = -gt + u\ln\left(\frac{m_0}{m}\right), \tag{7.56}$$

with m_0 the initial mass of the rocket. We should also note that since the burn rate is assumed constant, it must also be true that the mass loss is constant. That is,

$$\frac{dm}{dt} = -\alpha < 0, \tag{7.57}$$

or with a simple time integration

$$m_0 - m = \alpha t. \tag{7.58}$$

We can use equation (7.58) to substitute for t in equation (7.56) to express the speed of the rocket as a function of its mass only

$$v = -\frac{g}{\alpha}(m_0 - m) + u\ln\left(\frac{m_0}{m}\right). \tag{7.59}$$